Advanced Network Analysis

A Local Structure Graph Model

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Readings

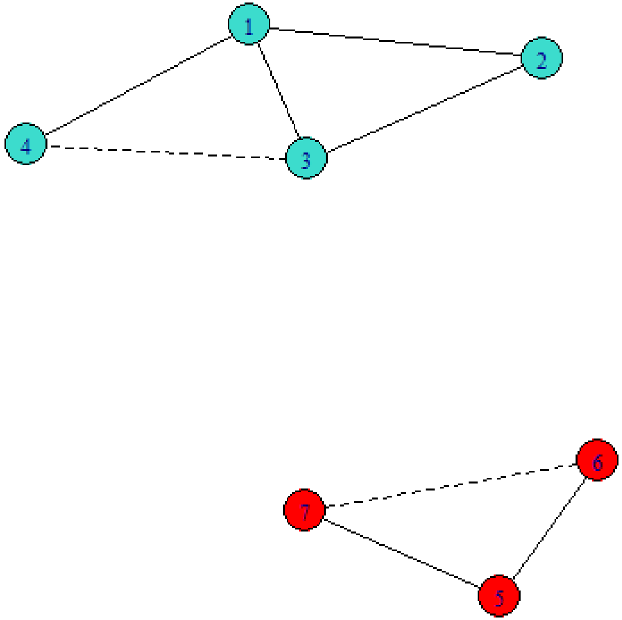
- Emily Casleton, Daniel Nordman, and Mark S. Kaiser. A local structure model for network analysis. *Statistics and Its Interface*, 2(10), 2017.
- Olga V. Chyzh and Mark S. Kaiser. Network analysis using a local structure graph model. *Political Analysis*, 27(4):397{414, 2019.

Motivation

- Do network edges form in reaction to (anticipation) of one another?
    - Did your ex start dating so soon because she didn't want you to start dating before her?
    - Do countries form alliances in response to their rivals or allies doing the same?
    - Do political parties form coalitions to balance against other (potential) coalitions?

Formal Motivation

- Do edges among red nodes affect the formation of edges among turquoise nodes?
- If nodes 6 and 7 formed and edge, would that make it more likely that nodes 3 and 4 form an edge?



The Estimator

- Suppose i is a potential edge in a network of potential edges (realized and unrealized);
- Then $s_i = (u_i, v_i)$ is i's location in Cartesian space.
- Denote the binary random variable, $y(s_i) = y_i$, so that:

$$y(s_i) = \begin{cases} 1 & \text{if edge } s_i \text{ is present} \\ 0 & \text{if edge } s_i \text{ is absent.} \end{cases}$$

- Define i's neighbors as $N_i = \{s_j : s_j \text{ is a neighbor of } s_i\}$.
- Make a Markov assumption of conditional spatial independence:

$$f(y(s_i)|\mathbf{y}(s_j) : s_j \neq s_i) = f(y(s_i)|\mathbf{y}(N_i))$$

- If connectivities between edges are continuous, then the Markov assumption is redundant.

The DV

```
##       edge_id Y  
##  [1,] "12"    "1"
##  [2,] "13"    "1"
##  [3,] "14"    "1"
##  [4,] "15"    "0"
##  [5,] "16"    "0"
##  [6,] "17"    "0"
##  [7,] "21"    "0"
##  [8,] "23"    "1"
##  [9,] "24"    "0"
## [10,] "25"    "0"
```

Set Up the Connectivity Matrix, W

- Start with an adjacency matrix among all potential edge-pairs.
- Code two edges as connected if they connect opposite-colored pairs of same-colored nodes.

	54	56	57	61	62
12	0	1	1	0	0
13	0	1	1	0	0
14	0	1	1	0	0
15	0	0	0	0	0
16	0	0	0	0	0

Set Up the Connectivity Matrix, W

```
W[1:10, 25:35]
```

```
##    51 52 53 54 56 57 61 62 63 64 65
## 12  0  0  0  0  1  1  0  0  0  0  1
## 13  0  0  0  0  1  1  0  0  0  0  1
## 14  0  0  0  0  1  1  0  0  0  0  1
## 15  0  0  0  0  0  0  0  0  0  0  0
## 16  0  0  0  0  0  0  0  0  0  0  0
## 17  0  0  0  0  0  0  0  0  0  0  0
## 21  0  0  0  0  1  1  0  0  0  0  1
## 23  0  0  0  0  1  1  0  0  0  0  1
## 24  0  0  0  0  1  1  0  0  0  0  1
## 25  0  0  0  0  0  0  0  0  0  0  0
```

The Binary Conditional Distribution

$$\mathrm{P}(Y_i = y_i|\boldsymbol{y}(N_i)) = \exp[A_i(\boldsymbol{y}(N_i))y_i - B_i(\boldsymbol{y}(N_i))], \tag{1}$$

where A_i is a natural parameter function and $B_i = \log[1 + \exp(A_i(y(N_i)))]$, and $\mathbf{y}\ (N_i)$ is a vector of values of the binary random variables (edges) of i's neighbors.

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Remember that the exponent of a difference is the ratio of exponents, $\exp(A - B) = \frac{\exp(A)}{\exp(B)}$:

$$\begin{aligned}
\mathrm{P}(Y_i = y_i|\boldsymbol{y}(N_i)) &= \frac{\exp[A_i(\boldsymbol{y}(N_i))y_i]}{\exp[B_i(\boldsymbol{y}(N_i))]} \\
&= \frac{\exp[A_i(\boldsymbol{y}(N_i))y_i]}{\exp[\log[1 + \exp(A_i(\boldsymbol{y}(N_i))y_i]]} \\
&= \frac{\exp[A_i(\boldsymbol{y}(N_i))y_i]}{1 + \exp[A_i(\boldsymbol{y}(N_i))y_i]},
\end{aligned} \tag{1}$$

The Natural Parameter Function

$$A_i\left(\boldsymbol{y}\left(N_i\right)\right) = \log\left(\frac{\kappa_i}{1-\kappa_i}\right) + \eta \sum_{j \in N_i} w_{ij}\left(y_j - \kappa_j\right), \tag{2}$$

where $\log\left(\frac{\kappa_i}{1-\kappa_i}\right) = \boldsymbol{X}_i^T\boldsymbol{\beta}$, $\boldsymbol{X}_i$ is a column vector of k exogenous covariates, $\boldsymbol{\beta}$ is a k by 1 vector of estimation parameters, w_{ij} is the ij^{th} element of a matrix of connectivities among edges $\mathbf{W}$, and η is its parameter.

- When $y_j > \kappa_j$, then the dependence term makes a positive contribution to $A_i(\mathbf{y}(N_i))$---complementary processes;
- When $y_j < \kappa_j$, then the dependence term makes a negative contribution to $A_i(\mathbf{y}(N_i))$---substitution-type processes;
- Key condition: $w_{ij} = w_{ji}$.
- Model does not require (prohibits) row-standardization of $\mathbf{w}$.

Estimation

$$\log PL = \sum_i \{y_i \log(p_i) + (1 - y_i)\log(1 - p_i))\}, \tag{3}$$

where:

$$p_i = \frac{\exp[A_i(y(N_i))]}{1 + \exp[A_i(y(N_i))]} \tag{4}$$

Let's Program This

Logit:

```
loglik_logit<-function(par,Y){
  b0<-par[1] #starting value for
  X<-rep(1,length(Y)) #constant
  xbeta<-as.matrix(X)%*%b0 #natu
  kappa<-exp(xbeta)/(1+exp(xbeta
  L<-Y*log(kappa)+(1-Y)*(log(1-k
  ell= -sum(L) #sum log-likelihc
  cat("ell",ell, fill=TRUE) #pri
  return(ell)
}
```

LSGM:

```
loglik_lsgm<-function(par,Y,W){
b0<-par[1] #starting value for c
eta<-par[2] #starting value for
X<-rep(1,length(Y)) #constant
xbeta<-as.matrix(X)%*%b0 #natura
kappa<-exp(xbeta)/(1+exp(xbeta))
A_i=log(kappa/(1-kappa))+eta*W%*
p_i<- exp(A_i)/(1+exp(A_i)) #Eqr
PL<-Y*log(p_i)+(1-Y)*log(1-p_i)
ell <- -sum(PL) #sum log pseudol
cat("ell",ell, fill=TRUE) #print
return(ell)
}
```

Estimate

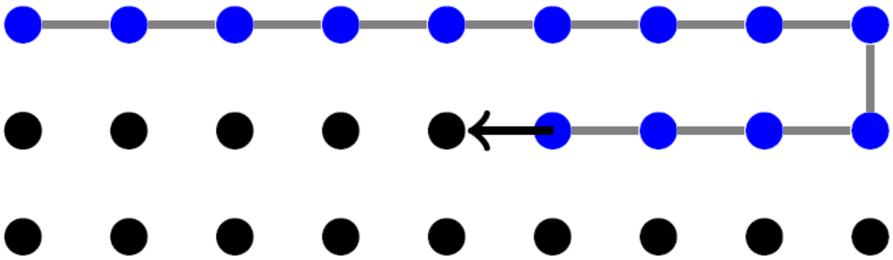
```
m1<-optim(par=c(0,0),loglik_lsgm,W=Wmat,Y=Y)
```

```
## ell 29.11218
## ell 30.66466
## ell 28.75713
## ell 27.40791
## ell 26.0759
## ell 25.96545
## ell 24.80895
## ell 22.84449
## ell 20.74303
## ell 20.01859
## ell 18.48442
## ell 18.40414
## ell 24.13454
## ell 24.70631
## ell 18.67434
## ell 20.15902
## ell 18.14229
## ell 17.96747
## ell 18.31736
## ell 17.78908
## ell 18.29955
```

Estimating Standard Errors

Need to simulate Y based on our parameter estimates:

- Start from a vector of initial $y_0 = \{y_{01}, y_{02}, \ldots, y_{0n}\}$ drawn from a binomial distribution;
- Moving row-wise, for i = 1, ..., n, individually simulate/update each observation as a function of previously simulated observations:



- n individual updates provide 1 full Gibbs iteration
- Use the result of the first Gibbs iteration as the new initial values and repeat to obtain M Gibbs networks y (can burn-in, thin, etc.).

Simulate Networks Based on m1

1. Function *spatbin.genone* simulates/updates a value for y for a single observation--step 2

2. Function *spatbin.onegibbs* applies *spatbin.genone* to update every observation of y--step 3

3. Function *spatbin.genfield* applies *spatbin.onegibbs* to generate M networks --step 4

```
spatbin.genone<-function(coeffs,w,curys){
b0<-coeffs[1]
eta<-coeffs[2]
X<-rep(1,length(Y))
xbeta<-as.matrix(X)%*%b0
kappa<-exp(xbeta)/(1+exp(xbeta))
A_i=log(kappa/(1-kappa))+eta*w%*%(curys-kappa)
p_i<- exp(A_i)/(1+exp(A_i))
y<- rbinom(n=length(curys), size=1, prob=p_i)
return(y)
}
```

```
spatbin.onegibbs<-function(coeffs,w,curys){
cnt<-0
n<-length(curys)
newys<-NULL
repeat{
    cnt<-cnt+1
    ny<-spatbin.genone(coeffs=coeffs,w=w,curys=curys)
    curys[cnt]<-ny[cnt]
    if(cnt==n) break
    }
newys<-curys
return(newys)
}
```

```r
spatbin.genfield<-function(coeffs,w,y0s,M){
curys<-y0s
cnt<-0
res<-as.data.frame(y0s)
repeat{
    cnt<-cnt+1
    newys<-spatbin.onegibbs(coeffs=coeffs,w=w,curys=curys)
    curys<-newys
    res<-cbind(res,curys)
    if(cnt==M) break
    }

return(res)
}
```

```r
n<-length(Y)
y0s=rbinom(n=n, size=1, prob=.5)
sims<-spatbin.genfield(coeffs=m1$par,w=W,y0s=y0s,M=1000)
#Take every 10th simulated network, i.e. burnin=10, thinning=10
sims<-sims[,seq(from=10, to=ncol(sims),by=10)]
```

Obtaining Standard Errors

1. Estimate the model on simulated networks (after burnin and thinning);
2. The standard errors are the standard deviations of the estimated coefficients.

Obtaining Standard Errors

```
sim_est<-function(Y){
 res<-optim(par=m1$par,loglik_lsgm,W=W,Y=as.matrix(Y))
 return(c(res$par,res$convergence))
}

library(parallel)
sim_est<-do.call("rbind",mclapply(sims, sim_est))
```

```
## ell 10.85043
## ell 10.50826
## ell 11.39045
## ell 13.67989
## ell 10.80678
## ell 10.66645
## ell 10.60053
## ell 10.41032
## ell 10.57674
## ell 10.86296
## ell 10.47248
## ell 10.36538
## ell 10.35012
## ell 10.37762
## ell 10.34535
```

Application: International Alliances

```r
#Open the data:
data("ally_data")
ally_data$tot_trade<-log(ally_data$tot_trade+1)
ally_data<-ally_data[ally_data$year==2007,]
ally_data[1:5,]
```

```
##     ccode1 ccode2 edge defense mil_ratio tot_trade joint_dem year
## 62       2     20    1       1 0.7231990  2.653716         1 2007
## 75       2     31    2       1 0.9936821  2.206597         1 2007
## 227      2     42    5       1 0.9553047  2.331096         1 2007
## 268      2     51    6       1 0.9852507  2.215345         1 2007
## 305      2     52    7       1 0.9901861  2.335300         1 2007
```

```r
#Prepare W:
W2007 <- W
W2007[1:5,1:5]
```

```
##        edge_diff1   edge_diff2   edge_diff5   edge_diff6   edge_diff7
## [1,] 0.000000e+00 1.777676e-06 1.858189e-06 2.092406e-06 2.086917e-06
## [2,] 1.777676e-06 0.000000e+00 8.051235e-08 3.147301e-07 3.092406e-07
## [3,] 1.858189e-06 8.051235e-08 0.000000e+00 2.342178e-07 2.287283e-07
## [4,] 2.092406e-06 3.147301e-07 2.342178e-07 0.000000e+00 5.489462e-09
```

Likelihood (1 X)

```
#Likelihood
loglik<-function(par,X,W,Y){
b0<-par[1]
b1<-par[2]
eta<-par[3]
xbeta<-b0+b1*X
kappa<-exp(xbeta)/(1+exp(xbeta)) #logit of Xb
A_i=log(kappa/(1-kappa))+eta*W%*%(Y-kappa) #Eqn 2
p_i<- exp(A_i)/(1+exp(A_i)) #Eqn 1, also Eqn 4
PL<-Y*log(p_i)+(1-Y)*log(1-p_i) #Eqn 3
ell <- -sum(PL)
#cat("ell",ell, fill=TRUE)
return(ell)
}
```

Let's Estimate

```
X=ally_data$tot_trade
Y=ally_data$defense
m1<-optim(par=c(0,0,0),loglik,X=X,W=W2007,Y=Y)
m1
```

```
## $par
## [1] -1.685218961 -0.007736929  0.937752856
##
## $value
## [1] 470.8573
##
## $counts
## function gradient
##       90       NA
##
## $convergence
## [1] 0
##
## $message
## NULL
```

Standard Errors

```
spatbin.genone<-function(coeffs,X,w,curys){
b0<-coeffs[1]
b1<-coeffs[2]
eta<-coeffs[3]
xbeta<- b0+b1*X
kappa<-exp(xbeta)/(1+exp(xbeta))
A_i=log(kappa/(1-kappa))+eta*w%*%(curys-kappa)
p_i<- exp(A_i)/(1+exp(A_i))
y<- rbinom(n=length(curys), size=1, prob=p_i)
return(y)
}
```

```r
spatbin.onegibbs<-function(coeffs,X,w,curys){
cnt<-0
n<-length(curys)
newys<-NULL
repeat{
    cnt<-cnt+1
    ny<-spatbin.genone(coeffs=coeffs,X=X,w=w,curys=curys)
    curys[cnt]<-ny[cnt]
    if(cnt==n) break
    }
newys<-curys
return(newys)
}
```

```r
spatbin.genfield<-function(coeffs,X,w,y0s,M){
curys<-y0s
cnt<-0
res<-as.data.frame(y0s)
repeat{
    cnt<-cnt+1
    newys<-spatbin.onegibbs(coeffs=coeffs,X=X,w=w,curys=curys)
    curys<-newys
    res<-cbind(res,curys)
    if(cnt==M) break
    }

return(res)
}
```

Simulate 1000 Random Networks

```
n<-length(Y)
y0s=rbinom(n=n, size=1, prob=.5)
sims<-spatbin.genfield(coeffs=m1$par,X=X,w=W2007,y0s=y0s,M=1000)
#Take every 10th simulated network, i.e. burnin=10, thinning=10
sims<-sims[,seq(from=10, to=ncol(sims),by=10)]
#saveRDS(sims, "sims.rds")
```

Estimate an LSGM on Each of the Simulated Networks

```
sims<-readRDS("data/sims.rds")
sim_est<-function(Y){
 res<-optim(par=m1$par,loglik,X=X,W=W2007,Y=as.matrix(Y))
 return(c(res$par,res$convergence))
}

library(parallel)
sim_est<-do.call("rbind",mclapply(sims, sim_est))
#Drop results if didn't converge (models that converged have converge
sim_est<-sim_est[sim_est[,4]==0,]
saveRDS(sim_est,"./data/sim_est.rds")
```

Calculate SEs and Make a Table

```
#Get sds of the estimates:
sim_est<-readRDS("data/sim_est.rds")
boot_se<-apply(sim_est,2,sd)
mytable<-cbind("coeff"=m1$par,"se"=boot_se[-4],"z-value"=(m1$par/boot
mytable
```

```
##              coeff        se     z-value
## [1,] -1.685218961 0.3132937 -5.37903845
## [2,] -0.007736929 0.1534057 -0.05043444
## [3,]  0.937752856 0.7400730  1.26710857
```

Package LSGM

Note: the package is in beta testing

```
library(devtools)
install_github("ochyzh/lsgm")
library(lsgm)
data(W)
data(toy_data)
lsgm(Y=as.matrix(toy_data$Y),W=W,X=as.data.frame(toy_data$X))
```

Your Turn

1. The above code estimates an lsgm with 1 edge-level covariate X using a pseudo-likelihood. Edit the code so that the resulting pseudo-likelihood can include 2 edge-level covariates $X1$ and $X2$ (i.e., the goal is to run an lsgm with *tot_trade* and *mil_ratio* as exogenous covariates).

2. Estimate an lsgm with *defense* as the dependent variable, with *tot_trade* and *mil_ratio* as the exogenous edge-level covariates, and *W* as the matrix that measure connectivities among defense edges.

3. Difficult: Change the code from the above slides to estimate the standard errors for your model.

Challenge Yourself: Insurgent Attacks

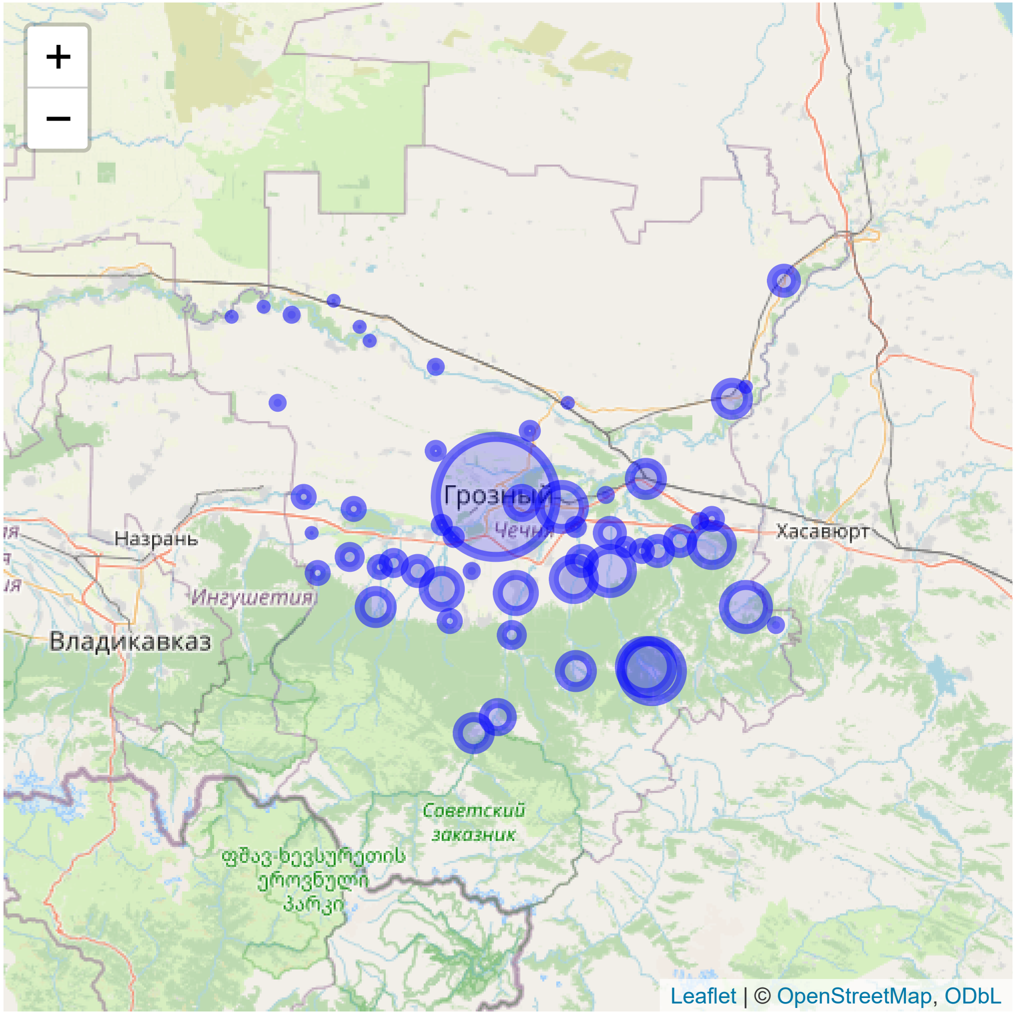
1. Download the *chechen_attacks* data that contains daily insurgent attacks in Chechnya and the *vilMat* matrix of distances among Chechen villages.

2. Use the *leaflet* package to visualize attacks by village.

3. Your goal is to estimate *attacks* as a function of other contemporaneous attacks, as well as population and a 7-day rolling sum of previous attacks. Because your data has a temporal component, you need to edit the likelihood to accommodate it.

```r
library(leaflet)
library(lubridate)
library(tidyverse)
chechen_attacks<- read.csv("./data/chechen_attacks.csv") |> dplyr::se
  dplyr::mutate(date=ymd(date)) |>
  dplyr::filter(date>="2000-03-08")

chechen_attacks |>
  group_by(village) |>
  summarise(attacks=sum(attack), lon=first(lon), lat=first(lat)) |>
  leaflet() |> addTiles() |>
  addCircleMarkers(~lon, ~lat, radius = ~2*sqrt(attacks), popup = ~at
```


+
−
Грозный
Чечня
Назрань
Хасавюрт
Ингушетия
Владикавказ
Советский
заказник
ფშავ-ხევსურეთის
ეროვნული
პარკი
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